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On an integral transform of the Mellin type

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SUMMARY

This paper establishes an inversion formula for an integral transform of the Mellin type which is defined on a truncated infinite interval $0 < a \leqslant r < \infty$ and which is associated with a radiation type boundary condition at $r = a$.

1. Introduction

There has been some some interest in the properties of two integral transforms which were proposed by the author [3] in 1963. These transforms, which are variations of the standard Mellin transform, are defined by the equations.

$$F_1(u) = \int_a^\infty (r^{u-1} - a^{2u} r^{-u-1}) f(r)\, dr, \tag{1}$$

$$F_2(u) = \int_a^\infty (r^{u-1} + a^{2u} r^{-u-1}) f(r)\, dr \tag{2}$$

where $a > 0$. Tweed [5]-[9] has shown how the above transforms can be used with effect in the solution of several dual and triple integral equations that arise in the theory of elasticity and convolution type formulas for the transforms have been developed by Harrington & Patel [1].

The transform (1) is useful in connection with boundary value problems involving the operator $r^2 f_{rr} + r f_r$ when the quantity $f(a)$ is prescribed, whereas the transform (2) is useful when the derivative $f'(a)$ rather than the function value $f(a)$ itself is assigned. In some problems a radiation type boundary condition appears in which the quantity $hf(a) + af'(a)$ occurs. A Mellin type transform associated with this boundary value has been proposed by Jain [2] who, in our notation, introduces the transform $F(u)$ defined by the equation

$$F(u) = \int_a^\infty [(u-h) r^{u-1} + (u+h) a^{2u} r^{-u-1}] f(r)\, dr. \tag{3}$$

In this formula h is a constant which may be complex.

Jain gives the following formula of inversion to be associated with (3);

$$f(r)=\frac{1}{2\pi i}\int_L \frac{r^{-u}F(u)du}{u-h}\,. \tag{4}$$

In this formula the path of integration L is a line parallel to the imaginary axis in the complex u-plane. The precise location of this line is not defined by Jain and will be determined later in this paper. The procedure adopted by Jain to derive the inversion formula (4) follows that used in the author's initial paper [3] where the transform (1) and the corresponding inversion formula were extracted from the solution of a boundary value problem formulated for a partial differential equation. This method generates the transforms (1) and (2) but is purely formal and does not constitute a proof of the corresponding formulas of inversion which can in fact be deduced from the standard Mellin inversion theorem.

This paper provides a rigorous investigation of the inversion formula associated with the transform defined by (3). It will be shown that the formula (4) proposed by Jain can be interpreted correctly provided that the point h lies in a certain half plane $Re(h)<\lambda$ and the path L is placed in a manner which will be defined. Here λ is a positive constant determined by the asymptotic behaviour of the function $f(r)$ as $r\to\infty$. However if $Re(h)>\lambda$ the formula (4) is in general incorrect no matter where L is located. In this event it will be shown that the proper inversion formula is given by the equation:

$$f(r)=r^{-h}F(h)+\frac{1}{2\pi i}\int_L \frac{r^{-u}F(u)du}{u-h} \tag{5}$$

where L is the line $Re(u)=c$ and $|c|<\lambda<Re(h)$.

It will be shown in the following section of this paper that, for functions $f(r)$ of the type considered, the transform $F(u)$ is analytic in the strip $|Re(u)|<\lambda$. If $Re(h)<\lambda$ the formula (4) is correct provided that the path L is placed in this strip and to the right of the point h. Since the path L in (4) is placed to the right of the pole at $u=h$ whereas that in (5) is placed to the left of this point, it is seen that (5) follows from (4), or vice versa, by moving L across the pole and taking into account the residue of the integrand at the pole. However, if equation (4) was rigorously established, (5) could not be deduced from (4) in this manner since, in general, the path L needs to be moved outside the domain of analyticity of $F(u)$, a procedure that could not be justified by Cauchy's theorem.

If $F(u)$ happens to be an entire function, for example when $f(r)$ vanished or is $O(e^{-\gamma r})$ for sufficiently large r, where $\gamma>0$, then formula (4) is valid for all h provided that L is positioned to the right of the pole.

2. The inversion formulas

The inversion formulas in question together with a set of conditions sufficient to ensure their validity are stated in the theorem that follows.

Theorem. Suppose that $f(\rho)$ is continuous for $\rho \geqslant a>0$ and of bounded variation in the neighbourhood of the point $\rho=r$ where $r>a$. Let $\rho^{\lambda-1}f(\rho)\in L\,(a,\infty)$ where $\lambda>0$ and let $F(u)$ be defined by equation (3) where h is a complex constant. Then

(i) if $Re(h) < \lambda$, $$f(r) = \frac{1}{2\pi i} \int_L \frac{r^{-u} F(u)du}{u-h} \tag{6}$$

where L is the line $(c-i\infty, c+i\infty)$ and $\max[-\lambda, Re(h)] < c < \lambda$;

(ii) if $Re(h) > \lambda$, $$f(r) = r^{-h}F(h) + \frac{1}{2\pi i} \int_L \frac{r^{-u} F(u)du}{u-h} \tag{7}$$

where L is the line $(c-i\infty, c+i\infty)$ and $|c| < \lambda$.

To establish the above theorem the transform $F(u)$ introduced by equation (3) will be expressed in terms of the truncated Mellin transforms defined by the equations

$$F_0(u) = \int_a^\infty r^{u-1} f(r)dr, \tag{8}$$

$$F_0(-u) = \int_a^\infty r^{-u-1} f(r)dr. \tag{9}$$

Then the equation (3) can be written as the equation

$$F(u) = (u-h)F_0(u) + (u+h)a^{2u} F_0(-u). \tag{10}$$

Since $r^{\lambda-1}f(r) \in L(a, \infty)$ it follows from equation (8) on writing $r^{u-1} = r^{u-\lambda+\lambda-1}$ that, on the line $Re(u) = t$,

$$|F_0(u)| \leqslant \int_a^\infty r^{t-\lambda+\lambda-1} \, |f(r)|\, dr \leqslant a^{t-\lambda} \int_a^\infty r^{\lambda-1} \, |f(r)|\, dr$$

provided that $t \leqslant \lambda$. It follows that the integral (8) is absolutely and uniformly convergent in any domain of values of u in the half plane $Re(u) \leqslant \lambda$ so that $F_0(u)$ is analytic in any such domain. In addition $F_0(u) = O(a^u)$ as $u \to \infty$ in this half plane. The function $F_0(-u)$ is then analytic in the corresponding half plane $Re(u) \geqslant -\lambda$ and is $O(a^{-u})$ as $u \to \infty$ in this half plane. In particular $F_0(u)$ and $F_0(-u)$ are analytic in the common strip $|Re(u)| \leqslant \lambda$ and so, by (10), $F(u)$ itself is analytic in this strip.

Upon applying the Mellin inversion theorem [4, p. 46] to equation (8) it follows that

$$\frac{1}{2\pi i} \int_L r^{-u} F_0(u)du = \begin{cases} f(r), & r > a \\ 0, & 0 < r < a \end{cases}, \tag{11}$$

where L is the line $Re(u) = c$ and $|c| < \lambda$.

The line L is now positioned in the strip $|Re(u)| < \lambda$ so that $|c| < \lambda$, and equation (10) rearranged to give the formula

$$F(u) = (u-h)F_0(u) + (u-h)\,a^{2u} F_0(-u) + 2h\,a^{2u} F_0(-u).$$

If the preceding equation is multiplied by $r^{-u}(u-h)^{-1}$ and integrated along L we obtain the equation

$$\int_L \frac{r^{-u}F(u)\,du}{u-h} = \int_L r^{-u}F_0(u)\,du + \int_L (a^2/r)^u F_0(-u)\,du + 2h \int_L \frac{r^{-u}a^{2u}F_0(-u)du}{u-h} \tag{12}$$

The existence of the integral appearing on the left hand side of equation (12) follows from that of the three integrals on the right hand side, the values of which will now be determined. The value of the first integral appearing on the right hand side of (12) is, by (11), equal to $2\pi\, if(r)$. The second integral occurring on the right hand side of (12) can be shown to be equal to zero by the following method. If the variable u be replaced by $-u$ in this integral it becomes equal to

$$\int_{L'} (a^2/r)^{-u} F_0(u)\,du \tag{13}$$

where L' denotes the path $Re(u) = -c$. Since $|c| < \lambda$ and $(a^2/r) < a$ it follows from the Mellin inversion formula (11) that the integral (13) is zero as required.

The third integral occurring on the right hand side of (12) can be evaluated by means of the calculus of residues as follows. The function $F_0(-u)$ is analytic in the half plane $Re(u) > -\lambda$ and is $O(a^{-u})$ as $u \to \infty$ in this half plane so that the integral in question can be evaluated by closing the contour on the right of L by means of a semicircle whose radius tends to infinity. On this semicircle the integral is $O[u^{-1}(a/r)^u]$ which, since $r > a$, tends to zero sufficiently rapidly as $u \to \infty$ to ensure that the contribution from the semicircle vanishes in the limit as the radius tends to infinity. The value of the integral then depends on whether the pole $u = h$ lies to the left or to the right of L.

If the given constant h is such that $Re(h) > \lambda$ then L is positioned to the left of the pole and the value of the third integral appearing on the right hand side of (12) is $-2\pi i\, r^{-h}a^{2h} F_0(-h)$ which, by (10), is equal to $-2\pi\, ir^{-h} F(h)$. On collecting these results it follows that equation (12) becomes

$$\int_L \frac{r^{-u}F(u)\,du}{u-h} = 2\pi i f(r) - 2\pi i r^{-h} F(h)$$

where L is the line $Re(u) = c$ and $|c| < \lambda < Re(h)$.

However if h is such that $Re(h) < \lambda$ and L, which is necessarily positioned such that $|c| < \lambda$, is chosen to ensure that $Re(h) < c$ then the pole lies to the left of L and the corresponding value of the integral is zero. Equation (12) then shows that

$$\int_L \frac{r^{-u}F(u)\,du}{u-h} = 2\pi i f(r)$$

where L is the line $Re(u) = c$ and $\max\,[-\lambda, Re(h)] < c < \lambda$.

3. Application

In applications it is advantageous to have at hand a formula for the transform of the group of terms $(r^2 f_{rr} + rf_r)$ that appears in the Laplace and other differential operators. For brevity we introduce the function $\psi(u,r)$ defined by the equation

$$\psi(u,r) = (u-h)r^u + (u+h)a^{2u} r^{-u} \tag{14}$$

so that

$$r^2 \psi_{rr} + r\psi_r - u^2 \psi = 0.$$

The transform $F(u)$ can then be defined by the equation

$$F(u) = \int_a^\infty \psi(u,r) f(r) \frac{dr}{r}. \tag{15}$$

The transform of the required group of terms is then given by the equation

$$\int_a^\infty (r^2 f_{rr} + rf_r)\, \psi(u,r) \frac{dr}{r} = u^2 F(u) - 2ua^u\, [hf(a) + af'(a)]. \tag{16}$$

This formula applies for $|Re(u)| < \lambda$ and can be established by repeated integration by parts.

4. A related transform

If the relevant interval of interest is $0 \leqslant r \leqslant a$ we introduce the related transform $H(u)$ defined by the equation

$$H(u) = \int_0^a \psi(u,r) f(r) \frac{dr}{r}. \tag{17}$$

In applying this transform it is assumed that $f(r)$ is continuous for $0 \leqslant r \leqslant a$ and that $r^{-\mu-1} f(r) \in L(0,a)$ where $\mu > 0$ so that $H(u)$ is regular in the strip $|Re(u)| < \mu$. By following a procedure similar to that adopted in Section 2 it can be shown that the inversion formulas to be associated with the transform $H(u)$ are given by the following equations:

(i) if $Re(h) < \mu$, $$f(r) = -r^{-h} H(h) + \frac{1}{2\pi i} \int_L \frac{r^{-u} H(u)\, du}{u-h} \tag{18}$$

where L is the line $(c-i\infty, c+i\infty)$ and $\max\,[-\mu, Re(h)] < c < \mu$;

(ii) if $Re(h) > \mu$,

$$f(r) = \frac{1}{2\pi i} \int_L \frac{r^{-u} H(u)\, du}{u-h} \tag{19}$$

where L is the line $(c-i\infty, c+i\infty)$ and $|c| < \mu$.

For the transform $H(u)$ it can be shown that the formula corresponding to (16) is given by the equation

$$\int_0^a (r^2 f_{rr} + r f_r)\,\psi(u,r)\,\frac{dr}{r} = u^2 H(u) + 2ua^u\,[hf(a) + af'(a)] \tag{20}$$

where $|Re(u)| < \mu$.

Appendix

In this appendix we consider the following pair of dual integral equations involving the transform $H(u)$ of the kind introduced in equation (17):

$$\frac{1}{2\pi i}\int_L \frac{r^{-u} H(u)\,du}{u-h} = 0, \quad 0 \leqslant r \leqslant b, \tag{21}$$

$$\frac{1}{2\pi i}\int_L \frac{u r^{-u-1} H(u) \cot u\lambda\, du}{u-h} = g'(r), \quad b \leqslant r \leqslant a. \tag{22}$$

These equations are typical of the type that arise when the Mellin type transforms are applied to the solution of mixed boundary value problems. When $h = 0$ the above equations reduce to those solved by Tweed in his paper [5]. The method adopted by Tweed can also be applied to solve (21), (22).

In the formulation of (21), (22) it is assumed that λ is a positive constant and that the path of integration is positioned to the left of the pole $u = h$. Then a comparison of (21) with (19) reveals that $H(u)$ is the transform of a function $f(r)$ that vanishes for $0 \leqslant r \leqslant b$ so that (17) reduces to the equation

$$H(u) = \int_b^a [(u-h) r^{u-1} + (u+h) a^{2u} r^{-u-1}] f(r)\,dr. \tag{23}$$

This expression automatically satisfies the condition (21) and shows that $H(u)$ is an entire function of the complex variable u. To proceed further it is first convenient to transform (23) by integration by parts. Since $f(b) = 0$ this procedure leads to the equation

$$H(u) = -\frac{1}{u}\int_b^a [2ha^u + (u-h) r^u - (u+h) a^{2u} r^{-u}] f'(r)\,dr.$$

This expression is now inserted into (22). After some manipulation, in which the resulting contour integral is evaluated by means of the calculus of residues, it is found that

$$\int_b^a f_1(\rho)\left[\frac{r^\gamma}{\rho^\gamma - a^\gamma} - \frac{r^\gamma \rho^\gamma}{a^{2\gamma} - r^\gamma \rho^\gamma}\right]\frac{d\rho}{\rho} = -\lambda g_1(r) \tag{24}$$

where $\gamma = \pi/\lambda$, $f_1(\rho) = hf(\rho) + \rho f'(\rho)$ and $g_1(r) = hg(r) + rg'(r)$, where $b \leqslant r \leqslant a$. The equation (24) is precisely the same as that solved by Tweed who reduced the equation to one of the Abel kind by suitable changes of variable. For details of the final solution of this equation we therefore refer to Tweed's paper.

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